

AN IWAHORI-MATSUMOTO PRESENTATION OF AFFINE YOKONUMA-HECKE ALGEBRAS

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Dedicated to Professor George Lusztig on his seventieth birthday

ABSTRACT. We first present an Iwahori-Matsumoto presentation of affine Yokonuma-Hecke algebras $\hat{Y}_{r,n}(q)$ to give a new proof of the fact, which was previously proved by Chlouveraki and Sécherre, that $\hat{Y}_{r,n}(q)$ is a particular case of the pro- p -Iwahori-Hecke algebras defined by Vignéras; meanwhile, we give one application. Using the new presentation, we then give a third presentation of $\hat{Y}_{r,n}(q)$, from which we immediately get an unexpected result, that is, the extended affine Hecke algebra of type A is a subalgebra of the affine Yokonuma-Hecke algebra.

1. INTRODUCTION

1.1. Affine Hecke algebras $\hat{\mathcal{H}}_n(q)$ were introduced by Iwahori and Matsumoto in [IM], in which they constructed an isomorphism between the convolution algebra $\mathbb{C}[I \backslash G(\mathbb{Q}_p) / I]$ of I -bi-invariant compactly supported functions on $G(\mathbb{Q}_p)$ and the specialization algebra $\hat{\mathcal{H}}_n(q)|_{q=p}$, where I is an Iwahori subgroup of $G(\mathbb{Q}_p)$. Later on, Bernstein found a totally different presentation of $\hat{\mathcal{H}}_n(q)$ in terms of an alternative set of generators and relations, which is a q -analogue of the presentation of the extended affine Weyl group, as a semi-direct product of the finite Weyl group and a lattice of translations. The representation theory of affine Hecke algebras is very important, and has been studied extensively over the past few decades; see [KL, CG, Xi] and so on.

1.2. When considering the convolution algebra $\mathbb{C}[I(1) \backslash GL_n(F) / I(1)]$ of compactly supported functions on $GL_n(F)$, where F is a local non-Archimedean field and $I(1)$ is the pro- p -radical of an Iwahori subgroup, Vignéras [Vi1] introduced the pro- p -Iwahori-Hecke algebras. In a recent series of papers [Vi2-4], Vignéras defined and studied the pro- p -Iwahori-Hecke algebras associated to p -adic reductive groups of arbitrary type. In particular, she gave the Iwahori-Matsumoto presentation and Bernstein presentation of them, described their centers, and classified their supersingular simple modules among other things. Pro- p -Iwahori-Hecke algebras play an important role in the study of mod- p representations of p -adic reductive groups.

1.3. Yokonuma-Hecke algebras were introduced by Yokonuma [Yo] as a centralizer algebra associated to the permutation representation of a finite Chevalley group G with respect to a maximal unipotent subgroup of G . In order to study the representations of Yokonuma-Hecke algebras, Chlouveraki and Poulain d'Andecy [ChPA] defined and studied affine Yokonuma-Hecke algebras $\hat{Y}_{r,n}(q)$. When q^2 is a power of a prime number p and

$r = q^2 - 1$, one can verify that $\widehat{Y}_{r,n}(q)$ is isomorphic to the specialized pro- p -Iwahori-Hecke algebra associated to $GL_n(F)$. Later on, Chlouveraki and Sécherre [ChS] proved that the affine Yokonuma-Hecke algebra is a particular case of the pro- p -Iwahori-Hecke algebras by using their Bernstein presentations.

In [CW], we gave the classification of the simple $\widehat{Y}_{r,n}(q)$ -modules as well as the classification of the simple modules of the cyclotomic Yokonuma-Hecke algebras over an algebraically closed field \mathbb{K} of characteristic p such that p does not divide r . Moreover, We [C1] and also Poulain d'Andecy [PA] proved that the affine Yokonuma-Hecke algebra is in fact isomorphic to a direct sum of matrix algebras over tensor products of extended affine Hecke algebras of type A . Recently, we [C2] established an explicit categorical equivalence between affine Yokonuma-Hecke algebras and quiver Hecke algebras associated to disjoint copies of quivers of (affine) type A , generalizing Rouquier's categorical equivalence theorem.

1.4. In order to give a Frobenius type formula for the characters of Ariki-Koike algebras, Shoji [S] first defined a variation of the Ariki-Koike algebra, called the modified Ariki-Koike algebra in [SS], as a way of approximating the usual Ariki-Koike algebra. In general the two algebras are not isomorphic, but they are isomorphic if a certain separation condition holds. Later on, Espinoza and Ryom-Hansen [ER] proved that the Yokonuma-Hecke algebra is isomorphic to the modified Ariki-Koike algebra. Thus, they gave a new proof of the isomorphism theorem for Yokonuma-Hecke algebras, previously proved by Lusztig [Lu2] and also by Jaco-Poulain d'Andecy [JaPA], by using the isomorphism theorem for modified Ariki-Koike algebras established by Sawada-Shoji [SS] and independently by Hu-Stoll [HS].

1.5. In this paper, in Section 2, we first present an Iwahori-Matsumoto presentation of affine Yokonuma-Hecke algebras $\widehat{Y}_{r,n}(q)$ to give a new proof of the fact, which was previously proved by Chlouveraki and Sécherre, that $\widehat{Y}_{r,n}(q)$ is a particular case of the pro- p -Iwahori-Hecke algebras defined by Vignéras; meanwhile, we develop one application, that is, we follow Jaco-Poulain d'Andecy's approach in [JaPA] to give a new proof of the isomorphism theorem for affine Yokonuma-Hecke algebras. In Section 3, we then introduce a third presentation of the affine Yokonuma-Hecke algebra using its Iwahori-Matsumoto presentation, from which we immediately obtain an unexpected result, that is, the extended affine Hecke algebra of type A is a subalgebra of the affine Yokonuma-Hecke algebra.

This is the first paper of a series. In two papers [C3] and [C4] which will come very soon, we shall define and study two types of affine Yokonuma-Schur algebras associated to the two presentations of affine Yokonuma-Hecke algebras given in this paper; for example, we shall define the standard bases and canonical bases of them and classify the simple modules of them among other things.

2. AN IWAHORI-MATSUMOTO PRESENTATION OF $\widehat{Y}_{r,n}(q)$

2.1. **Affine Yokonuma-Hecke algebras.** Let $r, n \in \mathbb{N}$, $r, n \geq 1$. Let q be an indeterminate and let $\mathcal{R} = \mathbb{Z}[\frac{1}{r}][q, q^{-1}]$.

Definition 2.1. The affine Yokonuma-Hecke algebra, denoted by $\widehat{Y}_{r,n} = \widehat{Y}_{r,n}(q)$, is an \mathcal{R} -associative algebra generated by the elements $\bar{t}_1, \dots, \bar{t}_n, g_1, \dots, g_{n-1}, X_1^{\pm 1}$, in which the generators $\bar{t}_1, \dots, \bar{t}_n, g_1, \dots, g_{n-1}$ satisfy the following relations:

$$g_i g_j = g_j g_i \quad \text{for all } i, j = 1, \dots, n-1 \text{ such that } |i-j| \geq 2; \quad (2.1)$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad \text{for all } i = 1, \dots, n-2; \quad (2.2)$$

$$\bar{t}_i \bar{t}_j = \bar{t}_j \bar{t}_i \quad \text{for all } i, j = 1, \dots, n; \quad (2.3)$$

$$g_i \bar{t}_j = \bar{t}_{s_i(j)} g_i \quad \text{for all } i = 1, \dots, n-1 \text{ and } j = 1, \dots, n; \quad (2.4)$$

$$\bar{t}_i^r = 1 \quad \text{for all } i = 1, \dots, n; \quad (2.5)$$

$$g_i^2 = 1 + (q - q^{-1}) \bar{e}_i g_i \quad \text{for all } i = 1, \dots, n-1; \quad (2.6)$$

where s_i is the transposition $(i, i+1)$, and for each $1 \leq i \leq n-1$,

$$\bar{e}_i := \frac{1}{r} \sum_{s=0}^{r-1} \bar{t}_i^s \bar{t}_{i+1}^{-s},$$

together with the following relations concerning the generators $X_1^{\pm 1}$:

$$X_1 X_1^{-1} = X_1^{-1} X_1 = 1; \quad (2.7)$$

$$g_1 X_1 g_1 X_1 = X_1 g_1 X_1 g_1; \quad (2.8)$$

$$g_i X_1 = X_1 g_i \quad \text{for all } i = 2, \dots, n-1; \quad (2.9)$$

$$\bar{t}_j X_1 = X_1 \bar{t}_j \quad \text{for all } j = 1, \dots, n. \quad (2.10)$$

By definition, we see that the elements \bar{e}_i 's are idempotents in $\widehat{Y}_{r,n}$, and the elements g_i 's are invertible with the inverse given by

$$g_i^{-1} = g_i - (q - q^{-1}) \bar{e}_i \quad \text{for all } i = 1, \dots, n-1. \quad (2.11)$$

For each $w \in \mathfrak{S}_n$, let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of w . By Matsumoto's lemma, the element $g_w := g_{i_1} g_{i_2} \cdots g_{i_r}$ does not depend on the choice of the reduced expression of w .

Let $i, k \in \{1, 2, \dots, n\}$ and set

$$\bar{e}_{i,k} := \frac{1}{r} \sum_{s=0}^{r-1} \bar{t}_i^s \bar{t}_k^{-s}. \quad (2.12)$$

Note that $\bar{e}_{i,i} = 1$, $\bar{e}_{i,k} = \bar{e}_{k,i}$, and that $\bar{e}_{i,i+1} = \bar{e}_i$. It can be easily checked that the following holds:

$$g_i \bar{e}_{j,k} = \bar{e}_{s_i(j), s_i(k)} g_i \quad \text{for } i = 1, \dots, n-1 \text{ and } j, k = 1, \dots, n. \quad (2.13)$$

In particular, we have $g_i \bar{e}_i = \bar{e}_i g_i$ for all $i = 1, \dots, n-1$.

We define the elements X_2, \dots, X_n in $\widehat{Y}_{r,n}$ by induction:

$$X_{i+1} := g_i X_i g_i \quad \text{for } i = 1, \dots, n-1. \quad (2.14)$$

Then it is proved in [ChPA, Lemma 1] that we have, for any $1 \leq i \leq n-1$,

$$g_i X_j = X_j g_i \quad \text{for } j = 1, 2, \dots, n \text{ such that } j \neq i, i+1. \quad (2.15)$$

Moreover, by [ChPA, Proposition 1], we have that the elements $\bar{t}_1, \dots, \bar{t}_n, X_1, \dots, X_n$ form a commutative family, that is,

$$xy = yx \quad \text{for any } x, y \in \{\bar{t}_1, \dots, \bar{t}_n, X_1, \dots, X_n\}. \quad (2.16)$$

2.2. An Iwahori-Matsumoto presentation. Let \widehat{W} be the extended affine Weyl group of type A , which is generated by $\rho, s_i, 0 \leq i \leq n-1$ satisfying the relations:

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \quad \text{if } i - j \not\equiv \pm 1 \pmod{n}; \quad (2.17)$$

$$\rho s_{\bar{i}} = s_{\bar{i}-1} \rho, \quad s_{\bar{i}} s_{\bar{i}+1} s_{\bar{i}} = s_{\bar{i}+1} s_{\bar{i}} s_{\bar{i}+1} \quad \text{for } 0 \leq i \leq n-1, \quad (2.18)$$

where $\bar{i} \in \{0, 1, \dots, n-1\}$ with $\bar{i} \equiv i \pmod{n}$.

Set $\mathfrak{X} := \mathbb{Z}^n$, which is identified with a free abelian group generated by X_1, \dots, X_n such that each element of \mathfrak{X} can be written in the form $X^\lambda := X_1^{\lambda_1} \dots X_n^{\lambda_n}$ for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$. The following lemma is well-known.

Lemma 2.2. *We have an isomorphism of groups $\widehat{W} \cong \mathfrak{X} \rtimes \mathfrak{S}_n$, which is given by*

$$\begin{aligned} s_i &\mapsto s_i \quad \text{for } 1 \leq i \leq n-1, \\ s_0 &\mapsto s_{n-1} \dots s_2 s_1 s_2 \dots s_{n-1} X_1 X_n^{-1}, \\ \rho &\mapsto s_{n-1} \dots s_1 X_1. \end{aligned} \quad (2.19)$$

Its inverse sends X_1 to $s_1 \dots s_{n-1} \rho$ and s_i to s_i for $1 \leq i \leq n-1$.

Let $\mathcal{T} = (\mathbb{Z}/r\mathbb{Z})^n$, which is a commutative group generated by $\bar{t}_1, \dots, \bar{t}_n$ with relations:

$$\begin{aligned} \bar{t}_i \bar{t}_j &= \bar{t}_j \bar{t}_i \quad \text{for all } i, j = 1, 2, \dots, n, \\ \bar{t}_i^r &= 1 \quad \text{for all } i = 1, 2, \dots, n. \end{aligned}$$

We can write each element of \mathcal{T} as $\bar{t}^\beta = \bar{t}_1^{\beta_1} \dots \bar{t}_n^{\beta_n}$ for $\beta = (\beta_1, \dots, \beta_n)$ with each $0 \leq \beta_i \leq r-1$. We consider the semi-direct product $\widehat{W}_{r,n} := (\mathcal{T} \times \mathfrak{X}) \rtimes \mathfrak{S}_n$, in which every element can be written as $\bar{t}^\beta X^\lambda \sigma$. Note that \bar{t}^β and X^λ commute with each other.

We also define a group $\widehat{W}_{r,n}$, which is generated by $t_j, 1 \leq j \leq n, \rho, s_i, 0 \leq i \leq n-1$ satisfying the following relations:

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \quad \text{if } i - j \not\equiv \pm 1 \pmod{n}; \quad (2.20)$$

$$\rho s_{\bar{i}} = s_{\bar{i}-1} \rho, \quad s_{\bar{i}} s_{\bar{i}+1} s_{\bar{i}} = s_{\bar{i}+1} s_{\bar{i}} s_{\bar{i}+1} \quad \text{for } 0 \leq i \leq n-1; \quad (2.21)$$

$$t_i^r = 1, \quad t_i t_j = t_j t_i \quad \text{for } 1 \leq i, j \leq n; \quad (2.22)$$

$$s_i t_j = t_{s_i(j)} s_i \quad \rho t_j = t_{j-1} \rho \quad \text{for } 1 \leq i \leq n-1, \quad (2.23)$$

where we set $t_0 := t_n$.

By generalizing Lemma 2.2, we can easily get the following result.

Lemma 2.3. *We have an isomorphism of groups $\widehat{W}_{r,n} \cong \widehat{W}_{r,n}$.*

Definition 2.4. We define an \mathcal{R} -associative algebra $\widehat{H}_{r,n}^{\text{aff}}$ generated by the elements $t_1, \dots, t_n, T_{s_0}, \dots, T_{s_{n-1}}, T_\rho^{\pm 1}$ with the following relations:

$$t_i^r = 1 \quad \text{for all } 1 \leq i \leq n; \quad (2.24)$$

$$t_i t_j = t_j t_i \quad \text{for all } 1 \leq i, j \leq n; \quad (2.25)$$

$$T_{s_i} t_j = t_{s_i(j)} T_{s_i} \quad \text{for all } 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n; \quad (2.26)$$

$$T_\rho t_j = t_{j-1} T_\rho \quad \text{for all } 1 \leq j \leq n; \quad (2.27)$$

$$T_\rho T_{s_{\bar{i}}} = T_{s_{\bar{i}-1}} T_\rho \quad \text{for all } 0 \leq i \leq n-1; \quad (2.28)$$

$$T_{s_i} T_{s_j} = T_{s_j} T_{s_i} \quad \text{if } i-j \not\equiv \pm 1 \pmod{n}; \quad (2.29)$$

$$T_{s_{\bar{i}}} T_{s_{\bar{i}+1}} T_{s_{\bar{i}}} = T_{s_{\bar{i}+1}} T_{s_{\bar{i}}} T_{s_{\bar{i}+1}} \quad \text{if } 0 \leq i \leq n-1 \text{ and } n \geq 3; \quad (2.30)$$

$$T_{s_i}^2 = 1 + (q - q^{-1}) e_i T_{s_i} \quad \text{for all } 0 \leq i \leq n-1; \quad (2.31)$$

$$T_\rho T_\rho^{-1} = T_\rho^{-1} T_\rho = 1, \quad (2.32)$$

where $t_0 := t_n$ and for each $0 \leq i \leq n-1$,

$$e_i := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_{i+1}^{-s}.$$

We now state the main result of this section, which can be regarded as a generalization of the isomorphism theorem between the Iwahori-Matsumoto presentation and Bernstein presentation of an extended affine Hecke algebra of type A.

Theorem 2.5. We have an \mathcal{R} -algebra isomorphism $\Phi : \widehat{H}_{r,n}^{\text{aff}} \rightarrow \widehat{Y}_{r,n}$ given by

$$\begin{aligned} \Phi : \quad & \begin{aligned} t_j &\longmapsto \bar{t}_j && \text{for } 1 \leq j \leq n, \\ T_{s_i} &\longmapsto g_i && \text{for } 1 \leq i \leq n-1, \\ T_{s_0} &\longmapsto X_1^{-1} X_n (g_{n-1} \cdots g_2 g_1 g_2 \cdots g_{n-1})^{-1}, \\ T_\rho &\longmapsto g_{n-1} \cdots g_1 X_1, \\ T_\rho^{-1} &\longmapsto X_1^{-1} g_1^{-1} \cdots g_{n-1}^{-1} \end{aligned} \end{aligned}$$

with the inverse $\Psi : \widehat{Y}_{r,n} \rightarrow \widehat{H}_{r,n}^{\text{aff}}$ defined by

$$\begin{aligned} \Psi : \quad & \begin{aligned} \bar{t}_j &\longmapsto t_j && \text{for } 1 \leq j \leq n, \\ g_i &\longmapsto T_{s_i} && \text{for } 1 \leq i \leq n-1, \\ X_1 &\longmapsto T_{s_1}^{-1} \cdots T_{s_{n-1}}^{-1} T_\rho, \\ X_1^{-1} &\longmapsto T_\rho^{-1} T_{s_{n-1}} \cdots T_{s_1}. \end{aligned} \end{aligned}$$

Proof. We extend Φ and Ψ defined on the generators to algebra homomorphisms. We need to show that Φ and Ψ preserve the defining relations of $\widehat{H}_{r,n}^{\text{aff}}$ and $\widehat{Y}_{r,n}$, respectively.

By (2.3)-(2.5), it is obvious that Φ preserves (2.24)-(2.26).

To show that Φ preserves (2.27), it suffices to prove that $g_{n-1} \cdots g_1 X_1 \bar{t}_j = \bar{t}_{j-1} g_{n-1} \cdots g_1 X_1$ for $1 \leq j \leq n$, where we set $\bar{t}_0 := \bar{t}_n$. It easily follows from (2.4) and (2.16).

To show that Φ preserves (2.28), it suffices to prove that

$$g_{n-1} \cdots g_1 X_1 \cdot \Phi(T_{s_{\bar{i}}}) = \Phi(T_{s_{\bar{i}-1}}) \cdot g_{n-1} \cdots g_1 X_1 \quad \text{for } 0 \leq i \leq n-1. \quad (2.33)$$

Note that

$$\begin{aligned}
\Phi(T_{s_0}) &= X_1^{-1} X_n g_{n-1}^{-1} \cdots g_2^{-1} g_1^{-1} g_2^{-1} \cdots g_{n-1}^{-1} \\
&= X_1^{-1} g_{n-1} \cdots g_2 g_1 X_1 g_2^{-1} \cdots g_{n-1}^{-1} \\
&= X_1^{-1} g_{n-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-1}^{-1} X_1.
\end{aligned} \tag{2.34}$$

For $i = 0$, in order to show (2.33), it suffices to prove

$$g_{n-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-1}^{-1} = g_1^{-1} g_2^{-1} \cdots g_{n-2}^{-1} g_{n-1} \cdots g_2 g_1. \tag{2.35}$$

By $g_{i+1} g_i g_{i+1}^{-1} = g_i^{-1} g_{i+1} g_i$ for $1 \leq i \leq n-2$, we have

$$\begin{aligned}
g_{n-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-1}^{-1} &= g_{n-1} \cdots g_3 g_1^{-1} g_2 g_1 g_3^{-1} \cdots g_{n-1}^{-1} \\
&= g_1^{-1} g_{n-1} \cdots g_4 g_2^{-1} g_3 g_2 g_4^{-1} \cdots g_{n-1}^{-1} g_1 \\
&= \cdots \\
&= g_1^{-1} g_2^{-1} \cdots g_{n-2}^{-1} g_{n-1} \cdots g_2 g_1.
\end{aligned}$$

For $i = 1$, in order to show (2.33), it suffices to prove

$$g_{n-1} \cdots g_2 g_1 X_1 g_1 X_1^{-1} g_1^{-1} \cdots g_{n-1}^{-1} = X_1^{-1} g_{n-1} \cdots g_2 g_1 X_1 g_2^{-1} \cdots g_{n-1}^{-1}, \tag{2.36}$$

which follows from (2.15) and the commutativity of $g_1 X_1 g_1$ and X_1^{-1} .

For $2 \leq i \leq n-1$, in order to show (2.33), it suffices to prove

$$g_{n-1} \cdots g_2 g_1 X_1 g_i = g_{i-1} g_{n-1} \cdots g_2 g_1 X_1. \tag{2.37}$$

Note that

$$\begin{aligned}
g_{i-1}^{-1} g_{n-1} \cdots g_2 g_1 X_1 g_i &= g_{n-1} \cdots g_{i+1} g_{i-1}^{-1} g_i g_{i-1} g_{i-2} \cdots g_1 X_1 g_i \\
&= g_{n-1} \cdots g_{i+1} g_i g_{i-1} g_i^{-1} g_{i-2} \cdots g_1 X_1 g_i \\
&= g_{n-1} \cdots g_{i+1} g_i g_{i-1} g_{i-2} \cdots g_1 g_i^{-1} X_1 g_i \\
&= g_{n-1} \cdots g_2 g_1 X_1.
\end{aligned}$$

We see that (2.37) holds.

To show that Φ preserves (2.29), it suffices to prove that $\Phi(T_{s_0})\Phi(T_{s_j}) = \Phi(T_{s_j})\Phi(T_{s_0})$ for $2 \leq j \leq n-2$, which is equivalent to the following identity:

$$g_j \cdot X_1^{-1} g_{n-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-1}^{-1} X_1 = X_1^{-1} g_{n-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-1}^{-1} X_1 \cdot g_j. \tag{2.38}$$

Note that

$$\begin{aligned}
&g_j \cdot X_1^{-1} g_{n-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-1}^{-1} X_1 \\
&= X_1^{-1} g_{n-1} \cdots g_{j+2} g_j g_{j+1} g_j g_{j-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-1}^{-1} X_1 \\
&= X_1^{-1} g_{n-1} \cdots g_{j+2} g_{j+1} g_j g_{j+1} g_{j-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-1}^{-1} X_1 \\
&= X_1^{-1} g_{n-1} \cdots g_{j+2} g_{j+1} g_j g_{j-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{j-1}^{-1} g_{j+1} g_j^{-1} g_{j+1} g_{j+2}^{-1} \cdots g_{n-1}^{-1} X_1 \\
&= X_1^{-1} g_{n-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{j-1}^{-1} g_j^{-1} g_{j+1}^{-1} g_j g_{j+2}^{-1} \cdots g_{n-1}^{-1} X_1 \\
&= X_1^{-1} g_{n-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-1}^{-1} X_1 \cdot g_j.
\end{aligned}$$

We see that (2.38) holds.

To show that Φ preserves (2.30), it suffices to prove that

$$\Phi(T_{s_1})\Phi(T_{s_0})\Phi(T_{s_1}) = \Phi(T_{s_0})\Phi(T_{s_1})\Phi(T_{s_0}) \quad (2.39)$$

and

$$\Phi(T_{s_{n-1}})\Phi(T_{s_0})\Phi(T_{s_{n-1}}) = \Phi(T_{s_0})\Phi(T_{s_{n-1}})\Phi(T_{s_0}). \quad (2.40)$$

We first show that (2.39) holds. We have

$$\begin{aligned} \Phi(T_{s_1})\Phi(T_{s_0})\Phi(T_{s_1}) &= g_1 \cdot X_1^{-1}X_n g_{n-1}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-1}^{-1} \cdot g_1 \\ &= g_1 \cdot X_1^{-1}X_n g_{n-1}^{-1} \cdots g_3^{-1}g_1^{-1}g_2^{-1}g_1^{-1}g_3^{-1} \cdots g_{n-1}^{-1} \cdot g_1 \\ &= g_1 X_1^{-1}X_n g_1^{-1} \cdot g_{n-1}^{-1} \cdots g_3^{-1}g_2^{-1}g_3^{-1} \cdots g_{n-1}^{-1}, \end{aligned} \quad (2.41)$$

and

$$\begin{aligned} &\Phi(T_{s_0})\Phi(T_{s_1})\Phi(T_{s_0}) \\ &= X_1^{-1}X_n g_{n-1}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-1}^{-1} \cdot g_1 \cdot X_1^{-1}X_n g_{n-1}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-1}^{-1} \\ &= X_1^{-1}X_n g_{n-1}^{-1} \cdots g_3^{-1}g_1^{-1}g_2^{-1}g_1^{-1}g_3^{-1} \cdots g_{n-1}^{-1} \cdot g_1 \cdot X_1^{-1}g_{n-1} \cdots g_2 g_1 X_1 g_2^{-1} \cdots g_{n-1}^{-1} \\ &= X_1^{-1}X_n g_{n-1}^{-1} \cdots g_3^{-1}g_1^{-1}g_2^{-1}g_3^{-1} \cdots g_{n-1}^{-1} \cdot X_1^{-1}g_{n-1} \cdots g_2 g_1 X_1 g_2^{-1} \cdots g_{n-1}^{-1} \\ &= X_1^{-1}X_n g_{n-1}^{-1} \cdots g_3^{-1}g_1^{-1}X_1^{-1}g_1 X_1 g_2^{-1} \cdots g_{n-1}^{-1} \\ &= X_1^{-1}X_n g_1^{-1}X_1^{-1}g_1 X_1 \cdot g_{n-1}^{-1} \cdots g_3^{-1}g_2^{-1} \cdots g_{n-1}^{-1} \end{aligned} \quad (2.42)$$

By (2.41) and (2.42), in order to show (2.39), it suffices to prove that

$$g_1 X_1^{-1}g_1^{-1} = X_1^{-1}g_1^{-1}X_1^{-1}g_1 X_1,$$

which follows from

$$g_1^{-1}X_1^{-1}g_1^{-1}X_1^{-1}g_1 X_1 g_1 = (g_1 X_1 g_1)^{-1} \cdot X_1^{-1} \cdot (g_1 X_1 g_1) = X_2^{-1}X_1^{-1}X_2 = X_1^{-1}.$$

Next we show that (2.40) holds. We have

$$\begin{aligned} \Phi(T_{s_{n-1}})\Phi(T_{s_0})\Phi(T_{s_{n-1}}) &= g_{n-1}X_1^{-1}X_n g_{n-1}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-1}^{-1}g_{n-1} \\ &= X_1^{-1}g_{n-1}X_n g_{n-1}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-1}^{-1}, \end{aligned} \quad (2.43)$$

and

$$\begin{aligned} &\Phi(T_{s_0})\Phi(T_{s_{n-1}})\Phi(T_{s_0}) \\ &= X_1^{-1}X_n g_{n-1}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-1}^{-1} \cdot g_{n-1} \cdot X_1^{-1}X_n g_{n-1}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-1}^{-1} \\ &= X_1^{-1}g_{n-1} \cdots g_2 g_1 X_1 g_2^{-1} \cdots g_{n-1}^{-1} \cdot X_1^{-1}X_n g_{n-1}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-1}^{-1} \\ &= X_1^{-1}g_{n-1}X_n g_{n-2} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-2}^{-1} \cdot g_{n-1}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-1}^{-1}. \end{aligned} \quad (2.44)$$

By (2.43) and (2.44), in order to show (2.40), it suffices to prove that

$$g_{n-1}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-2}^{-1} = g_{n-2} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-2}^{-1} \cdot g_{n-1}^{-1} \cdots g_1^{-1} \cdots g_{n-1}^{-1}. \quad (2.45)$$

By (2.2), we have

$$g_{n-2}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-2}^{-1} = g_1^{-1}g_2^{-1} \cdots g_{n-3}^{-1}g_{n-2}^{-1}g_{n-3}^{-1} \cdots g_2^{-1}g_1^{-1},$$

and

$$g_{n-1}^{-1} \cdots g_2^{-1}g_1^{-1}g_2^{-1} \cdots g_{n-1}^{-1} = g_1^{-1}g_2^{-1} \cdots g_{n-2}^{-1}g_{n-1}^{-1}g_{n-2}^{-1} \cdots g_2^{-1}g_1^{-1}.$$

Thus, in order to show (2.45), it suffices to prove that

$$g_{n-1}^{-1}g_1^{-1}g_2^{-1}\cdots g_{n-3}^{-1} = g_{n-2}\cdots g_2g_1g_2^{-1}\cdots g_{n-2}^{-1}\cdot g_1^{-1}\cdots g_{n-2}^{-1}g_{n-1}^{-1}, \quad (2.46)$$

that is,

$$g_1^{-1}g_2^{-1}\cdots g_{n-3}^{-1} = g_{n-2}\cdots g_2g_1g_2^{-1}\cdots g_{n-2}^{-1}\cdot g_1^{-1}\cdots g_{n-2}^{-1}, \quad (2.47)$$

which follows from (2.35).

By (2.34), we have

$$\begin{aligned} \Phi(T_{s_0})^2 &= X_1^{-1}g_{n-1}\cdots g_2g_1g_2^{-1}\cdots g_{n-1}^{-1}X_1 \cdot X_1^{-1}g_{n-1}\cdots g_2g_1g_2^{-1}\cdots g_{n-1}^{-1}X_1 \\ &= X_1^{-1}g_{n-1}\cdots g_2g_1^2g_2^{-1}\cdots g_{n-1}^{-1}X_1 \\ &= X_1^{-1}g_{n-1}\cdots g_2(1+(q-q^{-1})\bar{e}_1g_1)g_2^{-1}\cdots g_{n-1}^{-1}X_1 \\ &= 1+(q-q^{-1})\bar{e}_{n,1}X_1^{-1}g_{n-1}\cdots g_2g_1g_2^{-1}\cdots g_{n-1}^{-1}X_1 \\ &= 1+(q-q^{-1})\bar{e}_{n,1}\Phi(T_{s_0}), \end{aligned} \quad (2.48)$$

which shows that Φ preserves (2.31).

Next we show that Ψ preserve the defining relations of $\widehat{Y}_{r,n}$. It is obvious that Ψ preserve the relations (2.1)-(2.7).

In order to show that Ψ preserve (2.8), it suffices to prove that

$$\begin{aligned} T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_\rho \cdot T_{s_1} \cdot T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_\rho \cdot T_{s_1} \\ = T_{s_1} \cdot T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_\rho \cdot T_{s_1} \cdot T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_\rho. \end{aligned} \quad (2.49)$$

By (2.28), we have

$$T_\rho \cdot T_{s_2}^{-1}\cdots T_{s_{n-1}}^{-1}T_\rho \cdot T_{s_1} = T_{s_1}^{-1}\cdots T_{s_{n-2}}^{-1}T_\rho^2T_{s_1} = T_{s_1}^{-1}\cdots T_{s_{n-2}}^{-1}T_{s_{n-1}}T_\rho^2.$$

Thus, in order to prove (2.49), it suffices to show that

$$T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_{s_1}^{-1}\cdots T_{s_{n-2}}^{-1}T_{s_{n-1}} = T_{s_2}^{-1}\cdots T_{s_{n-1}}^{-1}T_{s_1}^{-1}\cdots T_{s_{n-2}}^{-1}, \quad (2.50)$$

that is,

$$T_{s_1}^{-1}\cdots T_{s_{n-2}}^{-1}T_{s_{n-1}}\cdots T_{s_1} = T_{s_{n-1}}\cdots T_{s_1} \cdot T_{s_2}^{-1}\cdots T_{s_{n-1}}^{-1}, \quad (2.51)$$

which follows from an argument similar to (2.35).

In order to show that Ψ preserve (2.9), it suffices to prove that

$$T_{s_i} \cdot T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_\rho = T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_\rho \cdot T_{s_i} \quad \text{for } 2 \leq i \leq n-1. \quad (2.52)$$

By (2.28), we have $T_\rho \cdot T_{s_i} = T_{s_{i-1}} \cdot T_\rho$ for $2 \leq i \leq n-1$. Thus, in order to show (2.52), it suffices to prove that

$$T_{s_i} \cdot T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1} = T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_{s_{i-1}}, \quad (2.53)$$

which follows from the identity $T_{s_i}T_{s_{i-1}}^{-1}T_{s_i}^{-1} = T_{s_{i-1}}^{-1}T_{s_i}^{-1}T_{s_{i-1}}$.

Finally we show that Ψ preserve (2.10). It suffices to prove that

$$T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_\rho \cdot t_j = t_j \cdot T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_\rho \quad \text{for } 1 \leq j \leq n. \quad (2.54)$$

For $j = 1$, we have $T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_\rho \cdot t_1 = T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}t_n \cdot T_\rho = \cdots = t_1T_{s_1}^{-1}\cdots T_{s_{n-1}}^{-1}T_\rho$.

For $2 \leq j \leq n$, we have

$$\begin{aligned} T_{s_1}^{-1} \cdots T_{s_{n-1}}^{-1} T_\rho \cdot t_j &= T_{s_1}^{-1} \cdots T_{s_{n-1}}^{-1} t_{j-1} \cdot T_\rho \\ &= T_{s_1}^{-1} \cdots T_{s_{j-2}}^{-1} t_j T_{s_{j-1}}^{-1} T_{s_j}^{-1} \cdot T_{s_{n-1}}^{-1} \cdot T_\rho \\ &= t_j \cdot T_{s_1}^{-1} \cdots T_{s_{n-1}}^{-1} T_\rho. \end{aligned}$$

We see that (2.54) holds.

It is obvious that $\Phi \circ \Psi = \text{Id}$ and $\Psi \circ \Phi = \text{Id}$. Thus, Φ and Ψ establish an isomorphism of algebras. \square

Let $S := \{s_1, \dots, s_{n-1}\}$ and $S^{\text{aff}} := \{s_0, s_1, \dots, s_{n-1}\}$. Let W^{aff} be the subgroup of \widehat{W} generated by s_0, s_1, \dots, s_{n-1} , which is exactly the affine Weyl group of type A . It is well-known that W^{aff} is a Coxeter group with a length function ℓ . We extend the length function ℓ from W^{aff} to \widehat{W} by letting $\ell(\rho^k w) = \ell(w)$ for any $k \in \mathbb{Z}$ and $w \in W^{\text{aff}}$. We further extend the length function ℓ from \widehat{W} to $\widehat{W}_{r,n}$ by setting $\ell(t\widehat{w}) = \ell(\widehat{w})$ for any $t \in \mathcal{T}$ and $\widehat{w} \in \widehat{W}$, where \mathcal{T} is identified with the subgroup of $\widehat{W}_{r,n}$ generated by the elements t_1, \dots, t_n by Lemma 2.3.

For each $\underline{w} \in \widehat{W}_{r,n}$, let $\underline{w} = t\rho^k s_{i_1} \cdots s_{i_r}$ be a reduced expression of \underline{w} . From the presentation of $\widehat{H}_{r,n}^{\text{aff}}$ given in Definition 2.4, we see that $T_{\underline{w}} := T_t T_\rho^k T_{s_{i_1}} \cdots T_{s_{i_r}}$ is well-defined, that is, it does not depend on the choice of the reduced expression of \underline{w} , and here we set $T_t := t \in \mathcal{T}$.

We first state the following lemma, which can be regarded as a generalization of [Ju, Lemmas 3 and 5].

Lemma 2.6. *In $\widehat{W}_{r,n}$, if there exist s_i, s_j ($0 \leq i, j \leq n-1$) and $\underline{w} \in \widehat{W}_{r,n}$ such that $\ell(s_i \underline{w} s_j) = \ell(\underline{w})$ and $\ell(s_i \underline{w}) = \ell(\underline{w} s_j)$, then we have*

- (1) $e_i s_i \underline{w} = e_i \underline{w} s_j$.
- (2) $s_i \underline{w} e_j = e_i \underline{w} s_j$.
- (3) $e_i \underline{w} = \underline{w} e_j$.

Proof. From the definition, we can get that $\rho e_i = e_{i-1} \rho$, $s_i e_i = e_i s_i$ and $s_i^t e_i = e_i s_i^t = e_i t$ for any $0 \leq i \leq n-1$.

(1) We assume that the reduced expression of \underline{w} is $\underline{w} = t\rho^k w$ for some $k \in \mathbb{Z}$ and $w \in W^{\text{aff}}$. By assumption, we have

$$\ell(s_i \underline{w} s_j) = \ell(s_i t \rho^k w s_j) = \ell(s_i^t \rho^k s_{i+k} w s_j) = \ell(s_{i+k} w s_j) = \ell(t \rho^k w) = \ell(w).$$

Similarly, we have

$$\ell(s_i \underline{w}) = \ell(s_i t \rho^k w) = \ell(s_i^t \rho^k s_{i+k} w) = \ell(s_{i+k} w) = \ell(t \rho^k w s_j) = \ell(w s_j).$$

By [Lu1, Proposition 1.10], we have $s_{i+k} w = w s_j$. Thus, we have

$$e_i s_i \underline{w} = e_i s_i t \rho^k w = e_i s_i^t \rho^k s_{i+k} w = e_i t \rho^k w s_j = e_i \underline{w} s_j.$$

(2) We prove it by induction on $\ell(\underline{w})$. If $\ell(\underline{w}) = 1$, then we assume that $\underline{w} = t\rho^h s_k$ for some $h \in \mathbb{Z}$ and $0 \leq k \leq n-1$.

If $k = i+h$, by the equality

$$\ell(s_i \underline{w}) = \ell(s_i t \rho^h s_k) = \ell(s_i^t \rho^h s_{i+h} s_k) = 0 = \ell(t \rho^h s_k s_j) = \ell(\underline{w} s_j),$$

we must have $k = j$. Thus, we get that

$$s_i \underline{w} e_j = s_i t \rho^h s_k e_j = {}^{s_i} t \rho^h s_{\overline{i+h}} s_k e_j = {}^{s_i} t \rho^h e_{\overline{i+h}} = {}^{s_i} t e_i \rho^h = e_i t \rho^h = e_i \underline{w} s_j.$$

If $k \neq \overline{i+h}$, by the equality

$$\ell(s_i \underline{w} s_j) = \ell(s_i t \rho^h s_k s_j) = \ell({}^{s_i} t \rho^h s_{\overline{i+h}} s_k s_j) = \ell(s_{\overline{i+h}} s_k s_j) = \ell(s_k) = \ell(\underline{w}),$$

we must have $j = \overline{i+h}$, and hence $k - \overline{i+h} \not\equiv \pm 1 \pmod{n}$. So we have $s_{\overline{i+h}} s_k = s_k s_{\overline{i+h}}$ and $s_k e_{\overline{i+h}} = e_{\overline{i+h}} s_k$. Thus, we get

$$\begin{aligned} s_i \underline{w} e_j &= s_i t \rho^h s_k e_j = {}^{s_i} t \rho^h s_{\overline{i+h}} s_k e_{\overline{i+h}} = {}^{s_i} t \rho^h e_{\overline{i+h}} s_{\overline{i+h}} s_k \\ &= {}^{s_i} t e_i \rho^h s_k s_{\overline{i+h}} = e_i t \rho^h s_k s_j = e_i \underline{w} s_j. \end{aligned}$$

Now we assume that the equality (2) is true if $\ell(\underline{w}) < n$. We suppose that $\underline{w} = t \rho^h s_{k_1} \cdots s_{k_n}$ is a reduced expression of \underline{w} .

If $\ell(\underline{w}) > \ell(s_i \underline{w})$, since we have $s_i \underline{w} = s_i t \rho^h s_{k_1} \cdots s_{k_n} = {}^{s_i} t \rho^h s_{\overline{i+h}} s_{k_1} \cdots s_{k_n}$, hence we get $\ell(s_{\overline{i+h}} s_{k_1} \cdots s_{k_n}) < \ell(s_{k_1} \cdots s_{k_n})$. By [Lu1, Proposition 1.7], we get that $s_{\overline{i+h}} s_{k_1} \cdots s_{k_n} = w'$ with $\ell(w') < n$, and so $s_{k_1} \cdots s_{k_n} = s_{\overline{i+h}} w'$.

We need to check that w' satisfies the equalities $\ell(s_{\overline{i+h}} w' s_j) = \ell(w')$ and $\ell(s_{\overline{i+h}} w') = \ell(w' s_j)$. By definition, we have $w' = \rho^{-h} \cdot {}^{s_i} t^{-1} s_i \underline{w}$ and $\rho^{-h} \cdot {}^{s_i} t^{-1} s_i = \rho^{-h} s_i t^{-1} = s_{\overline{i+h}} \rho^{-h} t^{-1}$. Thus, we get that

$$\ell(s_{\overline{i+h}} w' s_j) = \ell(s_{\overline{i+h}} \rho^{-h} \cdot {}^{s_i} t^{-1} s_i \underline{w} s_j) = \ell(\rho^{-h} t^{-1} \underline{w} s_j) = \ell(\underline{w} s_j) = \ell(s_i \underline{w}) = \ell(w'),$$

and

$$\ell(s_{\overline{i+h}} w') = \ell(s_{\overline{i+h}} \rho^{-h} \cdot {}^{s_i} t^{-1} s_i \underline{w}) = \ell(\rho^{-h} t^{-1} \underline{w}) = \ell(\underline{w}) = \ell(s_i \underline{w} s_j) = \ell(w' s_j).$$

So by induction, we get that $s_{\overline{i+h}} w' e_j = e_{\overline{i+h}} w' s_j$ and that

$$s_i \underline{w} e_j = s_i t \rho^h s_{\overline{i+h}} w' e_j = {}^{s_i} t \rho^h s_{\overline{i+h}} e_{\overline{i+h}} w' s_j = {}^{s_i} t e_i \rho^h s_{\overline{i+h}} w' s_j = e_i t \rho^h s_{\overline{i+h}} w' s_j = e_i \underline{w} s_j.$$

If $\ell(\underline{w}) < \ell(s_i \underline{w})$, then we have

$$\ell(s_i \cdot \underline{w} s_j \cdot s_j) = \ell(s_i \underline{w}) = \ell(\underline{w} s_j),$$

$$\ell(s_i \cdot \underline{w} s_j) = \ell(\underline{w}) = \ell(\underline{w} s_j \cdot s_j),$$

and

$$\ell(\underline{w} s_j) = \ell(s_i \cdot \underline{w}) > \ell(\underline{w}) = \ell(s_i \cdot \underline{w} s_j).$$

Thus, we can apply the first case on the element $\underline{w} s_j$, and get that $s_i(\underline{w} s_j) e_j = e_i(\underline{w} s_j) s_j$, that is, $s_i \underline{w} e_j = e_i \underline{w} s_j$.

(3) The equality follows directly from (1) and (2). \square

The following proposition can be proved by a standard argument (see [Lu1, Proposition 3.3] for instance).

Proposition 2.7. $\widehat{H}_{r,n}^{\text{aff}}$ has an \mathcal{R} -basis consisting of the following elements:

$$\{T_{\underline{w}} \mid \underline{w} \in \widehat{W}_{r,n}\}, \quad (2.55)$$

$$\text{or} \quad \{t_1^{\beta_1} \cdots t_n^{\beta_n} T_{\widehat{w}} \mid 0 \leq \beta_1, \dots, \beta_n \leq r-1, \widehat{w} \in \widehat{W}\}.$$

Proof. From Definition 2.4, it is obvious that we have

$$t_k T_{\underline{w}} = T_{t_k \underline{w}} \quad \text{for any } 1 \leq k \leq n.$$

For any $0 \leq i \leq n-1$, we have

$$T_{s_i} T_{\underline{w}} = \begin{cases} T_{s_i \underline{w}} & \text{if } \ell(s_i \underline{w}) = \ell(\underline{w}) + 1, \\ T_{s_i \underline{w}} + (q - q^{-1}) T_{e_i \underline{w}} & \text{if } \ell(s_i \underline{w}) = \ell(\underline{w}) - 1, \end{cases}$$

and

$$T_{\rho}^{\pm 1} T_{\underline{w}} = T_{\rho^{\pm 1} \underline{w}}.$$

Thus, the \mathcal{R} -submodule of $\widehat{H}_{r,n}^{\text{aff}}$ generated by $\{T_{\underline{w}} \mid \underline{w} \in \widehat{W}_{r,n}\}$ is a left ideal of $\widehat{H}_{r,n}^{\text{aff}}$. Since it contains $1 = T_1$, it is the whole algebra $\widehat{H}_{r,n}^{\text{aff}}$. In particular, $\widehat{H}_{r,n}^{\text{aff}}$ is generated by the elements $\{T_{\underline{w}} \mid \underline{w} \in \widehat{W}_{r,n}\}$.

Next we prove that the set $\{T_{\underline{w}} \mid \underline{w} \in \widehat{W}_{r,n}\}$ is an \mathcal{R} -basis of $\widehat{H}_{r,n}^{\text{aff}}$. Consider the free \mathcal{R} -module \mathcal{E} with bases $(e_{\underline{w}})_{\underline{w} \in \widehat{W}_{r,n}}$. For each $1 \leq k \leq n$ and $0 \leq i \leq n-1$, we define the following \mathcal{R} -linear maps from \mathcal{E} to \mathcal{E} by

$$\begin{aligned} P_{t_k}(e_{\underline{w}}) &= e_{t_k \underline{w}}, \\ P_{s_i}(e_{\underline{w}}) &= \begin{cases} e_{s_i \underline{w}} & \text{if } \ell(s_i \underline{w}) = \ell(\underline{w}) + 1, \\ e_{s_i \underline{w}} + (q - q^{-1}) e_{e_i \underline{w}} & \text{if } \ell(s_i \underline{w}) = \ell(\underline{w}) - 1, \end{cases} \\ P_{\rho}^{\pm 1}(e_{\underline{w}}) &= e_{\rho^{\pm 1} \underline{w}}, \end{aligned}$$

and

$$\begin{aligned} Q_{t_k}(e_{\underline{w}}) &= e_{\underline{w} t_k}, \\ Q_{s_i}(e_{\underline{w}}) &= \begin{cases} e_{\underline{w} s_i} & \text{if } \ell(\underline{w} s_i) = \ell(\underline{w}) + 1, \\ e_{\underline{w} s_i} + (q - q^{-1}) e_{\underline{w} e_i} & \text{if } \ell(\underline{w} s_i) = \ell(\underline{w}) - 1, \end{cases} \\ Q_{\rho}^{\pm 1}(e_{\underline{w}}) &= e_{\underline{w} \rho^{\pm 1}}. \end{aligned}$$

Set $\widehat{S}^{\text{aff}} := \{t_1, \dots, t_n\} \cup \{\rho^{\pm 1}\} \cup S^{\text{aff}}$. We first prove the following claim:

Claim(a): $P_u Q_v = Q_v P_u$ for any $u, v \in \widehat{S}^{\text{aff}}$.

When either u or v belongs to the set $\{t_1, \dots, t_n\} \cup \{\rho^{\pm 1}\}$, it is easy to check that **Claim(a)** holds. Thus, it suffices to check that $P_{s_i} Q_{s_j} = Q_{s_j} P_{s_i}$ for any $s_i, s_j \in S^{\text{aff}}$. This can be proved by distinguishing the following six cases. Let $\underline{w} \in \widehat{W}_{r,n}$.

Case 1. $s_i \underline{w} s_j, s_i \underline{w}, \underline{w} s_j, \underline{w}$ have lengths $q+2, q+1, q+1, q$. Then we have

$$P_{s_i} Q_{s_j}(e_{\underline{w}}) = Q_{s_j} P_{s_i}(e_{\underline{w}}) = e_{s_i \underline{w} s_j}.$$

Case 2. $\underline{w}, s_i \underline{w}, \underline{w} s_j, s_i \underline{w} s_j$ have lengths $q+2, q+1, q+1, q$. Then we have

$$\begin{aligned} P_{s_i} Q_{s_j}(e_{\underline{w}}) &= Q_{s_j} P_{s_i}(e_{\underline{w}}) \\ &= e_{s_i \underline{w} s_j} + (q - q^{-1}) e_{s_i \underline{w} e_j} + (q - q^{-1}) e_{e_i \underline{w} s_j} + (q - q^{-1})^2 e_{e_i \underline{w} e_j}. \end{aligned}$$

Case 3. $\underline{ws}_j, s_i\underline{ws}_j, \underline{w}, s_i\underline{w}$ have lengths $q+2, q+1, q+1, q$. Then we have

$$P_{s_i}Q_{s_j}(e_{\underline{w}}) = Q_{s_j}P_{s_i}(e_{\underline{w}}) = e_{s_i\underline{ws}_j} + (q - q^{-1})e_{e_i\underline{ws}_j}.$$

Case 4. $s_i\underline{w}, s_i\underline{ws}_j, \underline{w}, \underline{ws}_j$ have lengths $q+2, q+1, q+1, q$. Then we have

$$P_{s_i}Q_{s_j}(e_{\underline{w}}) = Q_{s_j}P_{s_i}(e_{\underline{w}}) = e_{s_i\underline{ws}_j} + (q - q^{-1})e_{s_i\underline{we}_j}.$$

Case 5. $s_i\underline{ws}_j, \underline{w}, \underline{ws}_j, s_i\underline{w}$ have lengths $q+1, q+1, q, q$. Then we have

$$P_{s_i}Q_{s_j}(e_{\underline{w}}) = e_{s_i\underline{ws}_j} + (q - q^{-1})e_{s_i\underline{we}_j} + (q - q^{-1})^2e_{e_i\underline{we}_j},$$

and

$$Q_{s_j}P_{s_i}(e_{\underline{w}}) = e_{s_i\underline{ws}_j} + (q - q^{-1})e_{e_i\underline{ws}_j} + (q - q^{-1})^2e_{e_i\underline{we}_j}.$$

By Lemma 2.6(2), we have $s_i\underline{we}_j = e_i\underline{ws}_j$. Thus, we get that $P_{s_i}Q_{s_j}(e_{\underline{w}}) = Q_{s_j}P_{s_i}(e_{\underline{w}})$.

Case 6. $s_i\underline{w}, \underline{ws}_j, \underline{w}, s_i\underline{ws}_j$ have lengths $q+1, q+1, q, q$. Then we have

$$P_{s_i}Q_{s_j}(e_{\underline{w}}) = e_{s_i\underline{ws}_j} + (q - q^{-1})e_{e_i\underline{ws}_j},$$

and

$$Q_{s_j}P_{s_i}(e_{\underline{w}}) = e_{s_i\underline{ws}_j} + (q - q^{-1})e_{s_i\underline{we}_j}.$$

Hence, we also get that $P_{s_i}Q_{s_j}(e_{\underline{w}}) = Q_{s_j}P_{s_i}(e_{\underline{w}})$ by Lemma 2.6(2).

Thus, we have proved the **Claim(a)**. Then we can repeat the arguments as done in the proof of [Lu1, Proposition 3.3] to conclude that the set $\{T_{\underline{w}} \mid \underline{w} \in \widehat{W}_{r,n}\}$ is an \mathcal{R} -basis of $\widehat{H}_{r,n}^{\text{aff}}$. Since the procedure is routine, we shall skip the details. \square

Set $S^{\text{aff}}(1) := \{ts \mid t \in \mathcal{T} \text{ and } s \in S^{\text{aff}}\}$. Take $q_{ts_i} := 1$ and $c_{ts_i} := (q - q^{-1})te_i$ for all $0 \leq i \leq n-1$ and $t \in \mathcal{T}$. It has been proved in [ChS, Section 4] that q_{ts_i} 's and c_{ts_i} 's satisfy the conditions [Vi2, Theorem 2.4(A)(1)-(2)].

Multiplying two sides of (2.31) by $T_t T_{s_i}$ and noting that $T_{s_i}te_i = e_i T_t$, we get that

$$T_{ts_i}^2 = (ts_i)^2 + c_{ts_i}T_{ts_i} \quad \text{for all } 0 \leq i \leq n-1 \text{ and } t \in \mathcal{T}. \quad (2.56)$$

Hence, from the presentation of $\widehat{H}_{r,n}^{\text{aff}}$ given in Definition 2.4 and (2.56), we see that the \mathcal{R} -bases $\{T_{\underline{w}} \mid \underline{w} \in \widehat{W}_{r,n}\}$ of $\widehat{H}_{r,n}^{\text{aff}}$ satisfy the braid and quadratic relations in [Vi2, Theorem 2.4(B)].

Thus, from Theorem 2.5 we immediately get the following corollary, which was previously proved in [ChS, Theorem 4.1].

Corollary 2.8. *The affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}$ is a particular case of the pro- p -Iwahori-Hecke algebras.*

2.3. One application. let \mathbb{K} be an algebraically closed field of characteristic p such that p does not divide r . In this subsection, we shall consider the specializations over \mathbb{K} of various algebras $\widehat{Y}_{r,n}$, $\widehat{H}_{r,n}^{\text{aff}}$, and so on; moreover, we shall denote the specialization algebras with the same symbols.

We first review some constructions presented in [JaPA, Section 2]. Assume that $\{\zeta_1, \dots, \zeta_r\}$ is the set of all r -th roots of unity. A character χ of \mathcal{T} over \mathbb{K} is determined by the values $\chi(t_j) \in \{\zeta_1, \dots, \zeta_r\}$ for $1 \leq j \leq n$. We denote by $\text{Irr}(\mathcal{T})$ the set of characters of \mathcal{T} over \mathbb{K} .

By Lemma 2.2, we will identify \widehat{W} with $\mathfrak{X} \rtimes \mathfrak{S}_n$. We have a natural group homomorphism σ from \widehat{W} to \mathfrak{S}_n , which is defined by

$$\sigma(X_j) = 1 \quad \text{for } 1 \leq j \leq n \quad \text{and} \quad \sigma(s_i) = s_i \quad \text{for } 1 \leq i \leq n-1.$$

Moreover, we have an action of \mathfrak{S}_n on \mathcal{T} by permutations, which in turn induces an action of \mathfrak{S}_n on $\text{Irr}(\mathcal{T})$ given by

$$w(\chi)(t_i) = \chi(t_{w^{-1}(i)}) \quad \text{for all } w \in \mathfrak{S}_n, \chi \in \text{Irr}(\mathcal{T}) \text{ and } 1 \leq i \leq n.$$

Thus, we get an action of \widehat{W} on $\text{Irr}(\mathcal{T})$ by composing σ and the action of \mathfrak{S}_n on $\text{Irr}(\mathcal{T})$ defined above.

For each $\chi \in \text{Irr}(\mathcal{T})$, the primitive idempotent E_χ of \mathcal{T} associated to χ can be explicitly written as follows:

$$E_\chi = \prod_{1 \leq i \leq n} \left(\frac{1}{r} \sum_{0 \leq s \leq r-1} \chi(t_i)^s t_i^{-s} \right). \quad (2.57)$$

Then, the set $\{E_\chi \mid \chi \in \text{Irr}(\mathcal{T})\}$ forms a complete set of orthogonal idempotents, and is a \mathbb{K} -basis of $\mathbb{K}\mathcal{T}$, where we identify the group algebra $\mathbb{K}\mathcal{T}$ of \mathcal{T} over \mathbb{K} with the subalgebra of $\widehat{H}_{r,n}^{\text{aff}}$ generated by t_1, \dots, t_n .

Lemma 2.9. *The elements $\{E_\chi T_{\widehat{w}} \mid \chi \in \text{Irr}(\mathcal{T}) \text{ and } \widehat{w} \in \widehat{W}\}$ is a \mathbb{K} -basis of $\widehat{H}_{r,n}^{\text{aff}}$.*

Proof. By Proposition 2.7 and the claims above, we see that $\widehat{H}_{r,n}^{\text{aff}}$ is generated by the elements $\{E_\chi T_{\widehat{w}}\}$. Thus, it suffices to prove that they are linearly independent. If they are linearly dependent, that is, there exist some $a_{ij} \in \mathbb{K}$ such that

$$\sum_{i,j} a_{ij} E_{\chi_i} T_{\widehat{w}_j} = 0, \quad (2.58)$$

where a_{ij} are not all zero. We might as well assume that $a_{11} \neq 0$. Multiplying two sides of (2.58) by E_{χ_1} , we get that $\sum_j a_{1j} E_{\chi_1} T_{\widehat{w}_j} = 0$. But E_{χ_1} can be written as a linear combination of some $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$'s. Thus, by the equality above we can easily get that the elements $\{t_1^{\beta_1} \cdots t_n^{\beta_n} T_{\widehat{w}}\}$ are linearly dependent. This is a contradiction. We are done. \square

An r -composition of n , denoted by $\mu \models n$, is an r -tuple $\mu = (\mu_1, \dots, \mu_r) \in \mathbb{Z}_{\geq 0}^r$ such that $\sum_{1 \leq a \leq r} \mu_a = n$. Let $\mathcal{C}_{r,n}$ be the set of r -compositions of n . Assume that $\chi \in \text{Irr}(\mathcal{T})$. For $a \in \{1, \dots, r\}$, let μ_a be the cardinal of elements $j \in \{1, \dots, n\}$ such that $\chi(t_j) = \zeta_a$. Then the sequence $(\mu_1, \dots, \mu_r) \in \mathcal{C}_{r,n}$, and we denote it by $\text{Comp}(\chi)$.

For each $\mu \models n$, we define a particular character $\chi_1^\mu \in \text{Irr}(\mathcal{T})$ by

$$\begin{cases} \chi_1^\mu(t_1) &= \dots = \chi_1^\mu(t_{\mu_1}) &= \zeta_1, \\ \chi_1^\mu(t_{\mu_1+1}) &= \dots = \chi_1^\mu(t_{\mu_1+\mu_2}) &= \zeta_2, \\ \vdots & \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \\ \chi_1^\mu(t_{\mu_1+\dots+\mu_{r-1}+1}) &= \dots = \chi_1^\mu(t_n) &= \zeta_r. \end{cases} \quad (2.59)$$

Notice that $\text{Comp}(\chi_1^\mu) = \mu$. From (2.59), we see that the stabilizer of χ_1^μ under the action of \mathfrak{S}_n is the Young subgroup \mathfrak{S}^μ , which is defined to be $\mathfrak{S}_{\mu_1} \times \dots \times \mathfrak{S}_{\mu_r}$. Notice that there is a unique representative of minimal length in each left coset in $\mathfrak{S}_n / \mathfrak{S}^\mu$. We shall denote

these distinguished left coset representatives by $\{\pi_{1,\mu}, \dots, \pi_{m_\mu,\mu}\}$ by the convention that $\pi_{1,\mu} = 1$ and set $\chi_k^\mu := \pi_{k,\mu}(\chi_1^\mu)$ for $1 \leq k \leq m_\mu$.

For each $\mu \models n$, we set

$$E_\mu := \sum_{\text{Comp}(\chi)=\mu} E_\chi.$$

Then the set $\{E_\mu \mid \mu \in \mathcal{C}_{r,n}\}$ forms a complete set of pairwise orthogonal central idempotents in $\widehat{H}_{r,n}^{\text{aff}}$. In particular, we have the following decomposition of $\widehat{H}_{r,n}^{\text{aff}}$ into a direct sum of two-sided ideals:

$$\widehat{H}_{r,n}^{\text{aff}} = \bigoplus_{\text{Comp}(\chi)=\mu} E_\mu \widehat{H}_{r,n}^{\text{aff}}. \quad (2.60)$$

Moreover, the following elements

$$\{E_{\chi_k^\mu} T_{\widehat{w}} \mid 1 \leq k \leq m_\mu \text{ and } \widehat{w} \in \widehat{W}\}$$

form a \mathbb{K} -basis of $E_\mu \widehat{H}_{r,n}^{\text{aff}}$.

Let $\widehat{\mathcal{H}}_n$ denote the extended affine Hecke algebra of type A associated to \widehat{W} over \mathbb{K} , which is endowed with a standard basis $\{S_{\widehat{w}} \mid \widehat{w} \in \widehat{W}\}$. For each $\mu \models n$, we denote by \widehat{H}^μ the \mathbb{K} -subalgebra of \widehat{H}_n generated by the elements $\{S_{\widehat{w}} \mid \widehat{w} \in \widehat{\mathfrak{S}}^\mu\}$, where $\widehat{\mathfrak{S}}^\mu := \mathbb{Z}^n \rtimes \mathfrak{S}^\mu$ is the subgroup of \widehat{W} , which is exactly the stabilizer of χ_1^μ under the action of \widehat{W} . The algebra \widehat{H}^μ is naturally isomorphic to $\widehat{\mathcal{H}}_{\mu_1} \otimes \dots \otimes \widehat{\mathcal{H}}_{\mu_r}$.

The following lemma can easily be proved by a direct calculation.

Lemma 2.10. *Let $\mu \models n$. There exists an algebra isomorphism*

$$\phi_\mu : \widehat{H}^\mu \xrightarrow{\sim} E_{\chi_1^\mu} \widehat{H}_{r,n}^{\text{aff}} E_{\chi_1^\mu},$$

which is defined by $\phi_\mu(S_{\widehat{w}}) = E_{\chi_1^\mu} T_{\widehat{w}} E_{\chi_1^\mu}$ for any $\widehat{w} \in \widehat{\mathfrak{S}}^\mu$.

For each $\mu \models n$, let $\text{Mat}_{m_\mu}(\widehat{H}^\mu)$ be the algebra of matrices of size m_μ with coefficients in \widehat{H}^μ . We define a linear map

$$\Phi_\mu : E_\mu \widehat{H}_{r,n}^{\text{aff}} \rightarrow \text{Mat}_{m_\mu}(\widehat{H}^\mu)$$

by

$$\Phi_\mu(E_{\chi_k^\mu} T_{\widehat{w}}) = S_{\pi_{k,\mu}^{-1} \widehat{w} \pi_{j,\mu}} M_{k,j}, \quad (2.61)$$

where $j \in \{1, \dots, m_\mu\}$ is the unique number such that $\widehat{w}(\chi_j^\mu) = \chi_k^\mu$ for given k , and $M_{k,j}$ denotes the elementary matrix with 1 in the position (k, j) .

We also define a linear map

$$\Psi_\mu : \text{Mat}_{m_\mu}(\widehat{H}^\mu) \rightarrow E_\mu \widehat{H}_{r,n}^{\text{aff}}$$

by

$$\Psi_\mu((S_{\widehat{w}_{i,j}})_{1 \leq i,j \leq m_\mu}) = \sum_{1 \leq i,j \leq m_\mu} E_{\chi_i^\mu} T_{\pi_{i,\mu} \widehat{w}_{i,j} \pi_{j,\mu}^{-1}} E_{\chi_j^\mu} \quad (2.62)$$

for $\widehat{w}_{i,j} \in \widehat{\mathfrak{S}}^\mu$.

We define the linear maps $\Phi_{r,n} := \bigoplus_{\mu \in \mathcal{C}_{r,n}} \Phi_\mu$ and $\Psi_{r,n} := \bigoplus_{\mu \in \mathcal{C}_{r,n}} \Psi_\mu$. The following theorem can be proved in exactly the same way as in [JaPA, Theorem 3.1], and we skip the details.

Theorem 2.11. *For $\mu \models n$, the linear map Φ_μ is an isomorphism of algebras with the inverse map Ψ_μ . Accordingly, $\Phi_{r,n}$ and $\Psi_{r,n}$ establish an isomorphism of algebras between $\widehat{H}_{r,n}^{\text{aff}}$ and $\bigoplus_{\mu \in \mathcal{C}_{r,n}} \text{Mat}_{m_\mu}(\widehat{H}^\mu)$.*

Combining Theorems 2.5 and 2.11, we immediately get the following result, which was previously proved in [C1, Theorem 5.1] and also [PA, Theorem 3.1].

Theorem 2.12. *There is a canonical isomorphism between the affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}$ and $\bigoplus_{\mu \in \mathcal{C}_{r,n}} \text{Mat}_{m_\mu}(\widehat{H}^\mu)$.*

3. A THIRD PRESENTATION

Let $\zeta = e^{2\pi i/r}$ and let A be the square matrix of degree r whose ij -entry is equal to $a_{ij} = \zeta^{j(i-1)}$ for $1 \leq i, j \leq r$; i.e., A is the usual Vandermonde matrix. Let $\Delta = \det A$ is the Vandermonde determinant, that is, $\Delta = \prod_{1 \leq j < i \leq r} (\zeta^i - \zeta^j)$. We can write the inverse of A as $A^{-1} = \Delta^{-1}B$, where $B = (b_{ij}(\zeta))$ is the adjoint matrix of A .

For each $1 \leq i \leq r$, we define a polynomial $F_i(X) \in \mathbb{Z}[\zeta][X]$ by

$$F_i(X) := \sum_{1 \leq j \leq r} b_{ij}(\zeta) X^{j-1}.$$

Let $\mathfrak{R} = \mathbb{Z}[q, q^{-1}, \zeta, \Delta^{-1}]$, where q is an indeterminate. We first give the definition of an \mathfrak{R} -associative algebra $\widehat{\mathcal{C}}_{r,n}^{\text{aff}}$.

Definition 3.1. We define an \mathfrak{R} -associative algebra $\widehat{\mathcal{C}}_{r,n}^{\text{aff}}$, which is generated by the elements $w_1, \dots, w_n, h_{s_0}, \dots, h_{s_{n-1}}, h_\rho^{\pm 1}$ with the following relations:

$$w_i^r = 1 \quad \text{for all } 1 \leq i \leq n; \quad (3.1)$$

$$w_i w_j = w_j w_i \quad \text{for all } 1 \leq i, j \leq n; \quad (3.2)$$

$$h_\rho w_j = w_{j-1} h_\rho \quad \text{for all } 1 \leq j \leq n; \quad (3.3)$$

$$h_\rho h_{s_i} = h_{s_{i-1}} h_\rho \quad \text{for all } 0 \leq i \leq n-1; \quad (3.4)$$

$$h_{s_i} h_{s_j} = h_{s_j} h_{s_i} \quad \text{if } i - j \not\equiv \pm 1 \pmod{n}; \quad (3.5)$$

$$h_{s_i} h_{s_{i+1}} h_{s_i} = h_{s_{i+1}} h_{s_i} h_{s_{i+1}} \quad \text{if } 0 \leq i \leq n-1 \text{ and } n \geq 3; \quad (3.6)$$

$$h_{s_i}^2 = 1 + (q - q^{-1}) h_{s_i} \quad \text{for all } 0 \leq i \leq n-1; \quad (3.7)$$

$$h_{s_i} w_i = w_{i+1} h_{s_i} - \Delta^{-2} \sum_{c_1 < c_2} (\zeta^{c_2} - \zeta^{c_1}) (q - q^{-1}) F_{c_1}(w_i) F_{c_2}(w_{i+1}) \quad \text{for } 1 \leq i \leq n-1; \quad (3.8)$$

$$h_{s_i} w_{i+1} = w_i h_{s_i} + \Delta^{-2} \sum_{c_1 < c_2} (\zeta^{c_2} - \zeta^{c_1}) (q - q^{-1}) F_{c_1}(w_i) F_{c_2}(w_{i+1}) \quad \text{for } 1 \leq i \leq n-1; \quad (3.9)$$

$$h_{s_i} w_l = w_l h_{s_i} \quad \text{for all } l \neq i, i+1 \text{ and } 1 \leq i \leq n-1; \quad (3.10)$$

$$h_\rho h_\rho^{-1} = h_\rho^{-1} h_\rho = 1, \quad (3.11)$$

where $w_0 := w_n$ and in the expressions above, the sum is taken over all $1 \leq c_1, c_2 \leq r$ such that $c_1 < c_2$.

Let $R = \mathbb{Z}[\frac{1}{r}][q, q^{-1}, \zeta, \Delta^{-1}]$. We extend the algebras $\widehat{Y}_{r,n}$, $\widehat{H}_{r,n}^{\text{aff}}$ and $\widehat{\mathcal{C}}_{r,n}^{\text{aff}}$ from \mathfrak{R} and \mathfrak{R} to R , respectively. We shall denote the extension algebras by the same notations such that $\widehat{Y}_{r,n}$, $\widehat{H}_{r,n}^{\text{aff}}$ and $\widehat{\mathcal{C}}_{r,n}^{\text{aff}}$ are all defined on R in the rest of this section.

We now state the main result of this section.

Theorem 3.2. *We have an R -algebra isomorphism $\phi : \widehat{H}_{r,n}^{\text{aff}} \rightarrow \widehat{\mathcal{C}}_{r,n}^{\text{aff}}$ given as follows:*
for $1 \leq j \leq n$,

$$\phi(t_j) = w_j,$$

for $0 \leq i \leq n-1$,

$$\phi(T_{s_i}) = h_{s_i} - \Delta^{-2} (q - q^{-1}) \sum_{c_1 < c_2} F_{c_1}(w_i) F_{c_2}(w_{i+1}),$$

and

$$\phi(T_\rho^{\pm 1}) = h_\rho^{\pm 1}.$$

Moreover, its inverse $\psi : \widehat{\mathcal{C}}_{r,n}^{\text{aff}} \rightarrow \widehat{H}_{r,n}^{\text{aff}}$ is defined as follows:
for $1 \leq j \leq n$,

$$\psi(w_j) = t_j,$$

for $0 \leq i \leq n-1$,

$$\psi(h_{s_i}) = T_{s_i} + \Delta^{-2} (q - q^{-1}) \sum_{c_1 < c_2} F_{c_1}(t_i) F_{c_2}(t_{i+1}),$$

and

$$\psi(h_\rho^{\pm 1}) = T_\rho^{\pm 1}.$$

We denote by $\mathcal{H}_{r,n}$ the subalgebra of $\widehat{H}_{r,n}^{\text{aff}}$ generated by the elements $t_1, \dots, t_n, T_{s_1}, \dots, T_{s_{n-1}}$, which is canonically isomorphic to the Yokonuma-Hecke algebra defined in [ChPA, Section 2.1], and we shall not distinguish between them.

We denote by $\mathcal{C}_{r,n}$ the subalgebra of $\widehat{\mathcal{C}}_{r,n}^{\text{aff}}$ generated by the elements $w_1, \dots, w_n, h_{s_1}, \dots, h_{s_{n-1}}$, which is canonically isomorphic to the modified Ariki-Koike algebra defined in [S, Section 3.6] with $u_i = \zeta^i$ for $1 \leq i \leq r$, and we shall not distinguish between them.

The following proposition has been proved in [ER, Theorem 7].

Proposition 3.3. *There is a canonical isomorphism between $\mathcal{H}_{r,n}$ and $\mathcal{C}_{r,n}$, which is explicitly described by $\varphi : \mathcal{H}_{r,n} \rightarrow \mathcal{C}_{r,n}$ and $\chi : \mathcal{C}_{r,n} \rightarrow \mathcal{H}_{r,n}$, where φ is defined as follows: for $1 \leq j \leq n$,*

$$\varphi(t_j) = w_j,$$

for $1 \leq i \leq n-1$,

$$\varphi(T_{s_i}) = h_{s_i} - \Delta^{-2}(q - q^{-1}) \sum_{c_1 < c_2} F_{c_1}(w_i) F_{c_2}(w_{i+1}),$$

and its inverse χ is defined as follows:

for $1 \leq j \leq n$,

$$\chi(w_j) = t_j,$$

for $1 \leq i \leq n-1$,

$$\chi(h_{s_i}) = T_{s_i} + \Delta^{-2}(q - q^{-1}) \sum_{c_1 < c_2} F_{c_1}(t_i) F_{c_2}(t_{i+1}).$$

We then give the definition of two algebras as follows.

Definition 3.4. We denote by $\mathcal{H}_{r,n}^1$ the subalgebra of $\widehat{H}_{r,n}^{\text{aff}}$ generated by the elements $t_1, \dots, t_n, T_{s_0}, \dots, T_{s_{n-2}}$, whose explicit presentation is as follows:

$$t_i^r = 1 \quad \text{for all } 1 \leq i \leq n; \quad (3.12)$$

$$t_i t_j = t_j t_i \quad \text{for all } 1 \leq i, j \leq n; \quad (3.13)$$

$$T_{s_0} t_1 = t_n T_{s_0} \quad (3.14)$$

$$T_{s_0} t_n = t_1 T_{s_0} \quad (3.15)$$

$$T_{s_0} t_k = t_k T_{s_0} \quad \text{for all } 2 \leq k \leq n-1; \quad (3.16)$$

$$T_{s_i} t_j = t_{s_i(j)} T_{s_i} \quad \text{for all } 1 \leq i \leq n-2 \text{ and } 1 \leq j \leq n; \quad (3.17)$$

$$T_{s_i} T_{s_j} = T_{s_j} T_{s_i} \quad \text{if } |i - j| > 1; \quad (3.18)$$

$$T_{s_i} T_{s_{i+1}} T_{s_i} = T_{s_{i+1}} T_{s_i} T_{s_{i+1}} \quad \text{if } 0 \leq i \leq n-3 \text{ and } n \geq 3; \quad (3.19)$$

$$T_{s_i}^2 = 1 + (q - q^{-1}) e_i T_{s_i} \quad \text{for all } 0 \leq i \leq n-2, \quad (3.20)$$

where $t_0 := t_n$ and for each $0 \leq i \leq n-2$,

$$e_i := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_{i+1}^{-s}.$$

Definition 3.5. We denote by $\mathcal{C}_{r,n}^1$ the subalgebra of $\widehat{\mathcal{C}}_{r,n}^{\text{aff}}$ generated by the elements $w_1, \dots, w_n, h_{s_0}, \dots, h_{s_{n-2}}$, whose explicit presentation is as follows:

$$w_i^r = 1 \quad \text{for all } 1 \leq i \leq n; \quad (3.21)$$

$$w_i w_j = w_j w_i \quad \text{for all } 1 \leq i, j \leq n; \quad (3.22)$$

$$h_{s_i} h_{s_j} = h_{s_j} h_{s_i} \quad \text{if } |i - j| > 1; \quad (3.23)$$

$$h_{s_i} h_{s_{i+1}} h_{s_i} = h_{s_{i+1}} h_{s_i} h_{s_{i+1}} \quad \text{if } 0 \leq i \leq n-3 \text{ and } n \geq 3; \quad (3.24)$$

$$h_{s_i}^2 = 1 + (q - q^{-1}) h_{s_i} \quad \text{for all } 0 \leq i \leq n-2; \quad (3.25)$$

$$h_{s_i} w_i = w_{i+1} h_{s_i} - \Delta^{-2} \sum_{c_1 < c_2} (\zeta^{c_2} - \zeta^{c_1}) (q - q^{-1}) F_{c_1}(w_i) F_{c_2}(w_{i+1}) \quad \text{for } 0 \leq i \leq n-2; \quad (3.26)$$

$$h_{s_i} w_{i+1} = w_i h_{s_i} + \Delta^{-2} \sum_{c_1 < c_2} (\zeta^{c_2} - \zeta^{c_1}) (q - q^{-1}) F_{c_1}(w_i) F_{c_2}(w_{i+1}) \quad \text{for } 0 \leq i \leq n-2; \quad (3.27)$$

$$h_{s_i} w_l = w_l h_{s_i} \quad \text{for all } l \not\equiv i, i+1 \pmod{n} \text{ and } 0 \leq i \leq n-2, \quad (3.28)$$

where $w_0 := w_n$ and in the expressions above, the sum is taken over all $1 \leq c_1, c_2 \leq r$ such that $c_1 < c_2$.

By using Proposition 3.3, we can easily get the following result.

Proposition 3.6. *There is a canonical isomorphism between $\mathcal{H}_{r,n}^1$ and $\mathcal{C}_{r,n}^1$, which is explicitly described by $\phi_1 : \mathcal{H}_{r,n}^1 \rightarrow \mathcal{C}_{r,n}^1$ and $\psi_1 : \mathcal{C}_{r,n}^1 \rightarrow \mathcal{H}_{r,n}^1$, where ϕ_1 is defined as follows:*

for $1 \leq j \leq n$,

$$\phi_1(t_j) = w_j,$$

for $0 \leq i \leq n-2$,

$$\phi_1(T_{s_i}) = h_{s_i} - \Delta^{-2} (q - q^{-1}) \sum_{c_1 < c_2} F_{c_1}(w_i) F_{c_2}(w_{i+1}),$$

and its inverse ψ_1 is defined as follows:

for $1 \leq j \leq n$,

$$\psi_1(w_j) = t_j,$$

for $0 \leq i \leq n-2$,

$$\psi_1(h_{s_i}) = T_{s_i} + \Delta^{-2} (q - q^{-1}) \sum_{c_1 < c_2} F_{c_1}(t_i) F_{c_2}(t_{i+1}).$$

We further define the following two algebras.

Definition 3.7. We denote by $\mathcal{H}_{r,n}^2$ the subalgebra of $\widehat{H}_{r,n}^{\text{aff}}$ generated by the elements $t_1, \dots, t_n, T_{s_0}, T_{s_2}, \dots, T_{s_{n-1}}$, whose explicit presentation is as follows:

$$t_i^r = 1 \quad \text{for all } 1 \leq i \leq n; \quad (3.29)$$

$$t_i t_j = t_j t_i \quad \text{for all } 1 \leq i, j \leq n; \quad (3.30)$$

$$T_{s_0} t_1 = t_n T_{s_0} \quad (3.31)$$

$$T_{s_0} t_n = t_1 T_{s_0} \quad (3.32)$$

$$T_{s_0} t_k = t_k T_{s_0} \quad \text{for all } 2 \leq k \leq n-1; \quad (3.33)$$

$$T_{s_i} t_j = t_{s_i(j)} T_{s_i} \quad \text{for all } 2 \leq i \leq n-1 \text{ and } 1 \leq j \leq n; \quad (3.34)$$

$$T_{s_i} T_{s_j} = T_{s_j} T_{s_i} \quad \text{if } |i-j| > 1 \text{ and } 2 \leq i, j \leq n-1; \quad (3.35)$$

$$T_{s_i} T_{s_{i+1}} T_{s_i} = T_{s_{i+1}} T_{s_i} T_{s_{i+1}} \quad \text{if } 2 \leq i \leq n-2 \text{ and } n \geq 3; \quad (3.36)$$

$$T_{s_0} T_{s_k} = T_{s_k} T_{s_0} \quad \text{for all } 2 \leq k \leq n-2; \quad (3.37)$$

$$T_{s_0} T_{s_{n-1}} T_{s_0} = T_{s_{n-1}} T_{s_0} T_{s_{n-1}} \quad (3.38)$$

$$T_{s_i}^2 = 1 + (q - q^{-1}) e_i T_{s_i} \quad \text{for all } i = 0 \text{ and } 2 \leq i \leq n-1, \quad (3.39)$$

where $t_0 := t_n$ and for each $i = 0$ and $2 \leq i \leq n-1$,

$$e_i := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_{i+1}^{-s}.$$

Definition 3.8. We denote by $\mathcal{C}_{r,n}^2$ the subalgebra of $\widehat{\mathcal{C}}_{r,n}^{\text{aff}}$ generated by the elements $w_1, \dots, w_n, h_{s_0}, h_{s_2}, \dots, h_{s_{n-1}}$, whose explicit presentation is as follows:

$$w_i^r = 1 \quad \text{for all } 1 \leq i \leq n; \quad (3.40)$$

$$w_i w_j = w_j w_i \quad \text{for all } 1 \leq i, j \leq n; \quad (3.41)$$

$$h_{s_i} h_{s_j} = h_{s_j} h_{s_i} \quad \text{if } |i-j| > 1 \text{ and } 2 \leq i, j \leq n-1; \quad (3.42)$$

$$h_{s_i} h_{s_{i+1}} h_{s_i} = h_{s_{i+1}} h_{s_i} h_{s_{i+1}} \quad \text{if } 2 \leq i \leq n-2 \text{ and } n \geq 3; \quad (3.43)$$

$$h_{s_0} h_{s_k} = h_{s_k} h_{s_0} \quad \text{for all } 2 \leq k \leq n-2; \quad (3.44)$$

$$h_{s_0} h_{s_{n-1}} h_{s_0} = h_{s_{n-1}} h_{s_0} h_{s_{n-1}} \quad (3.45)$$

$$h_{s_i}^2 = 1 + (q - q^{-1}) h_{s_i} \quad \text{for all } i = 0 \text{ or } 2 \leq i \leq n-1; \quad (3.46)$$

$$h_{s_i} w_i = w_{i+1} h_{s_i} - \Delta^{-2} \sum_{c_1 < c_2} (\zeta^{c_2} - \zeta^{c_1}) (q - q^{-1}) F_{c_1}(w_i) F_{c_2}(w_{i+1}) \quad \text{for } i = 0 \text{ or } 2 \leq i \leq n-1; \quad (3.47)$$

$$h_{s_i} w_{i+1} = w_i h_{s_i} + \Delta^{-2} \sum_{c_1 < c_2} (\zeta^{c_2} - \zeta^{c_1}) (q - q^{-1}) F_{c_1}(w_i) F_{c_2}(w_{i+1}) \quad \text{for } i = 0 \text{ or } 2 \leq i \leq n-1; \quad (3.48)$$

$$h_{s_i} w_l = w_l h_{s_i} \quad \text{for all } l \not\equiv i, i+1 \pmod{n} \text{ and } i = 0 \text{ or } 2 \leq i \leq n-1, \quad (3.49)$$

where $w_0 := w_n$ and in the expressions above, the sum is taken over all $1 \leq c_1, c_2 \leq r$ such that $c_1 < c_2$.

By using Proposition 3.3 again, we can easily get the following result.

Proposition 3.9. *There is a canonical isomorphism between $\mathcal{H}_{r,n}^2$ and $\mathcal{C}_{r,n}^2$, which is explicitly described by $\phi_2 : \mathcal{H}_{r,n}^2 \rightarrow \mathcal{C}_{r,n}^2$ and $\psi_2 : \mathcal{C}_{r,n}^2 \rightarrow \mathcal{H}_{r,n}^2$, where ϕ_2 is defined as follows:*

for $1 \leq j \leq n$,

$$\phi_2(t_j) = w_j,$$

for $i = 0$ or $2 \leq i \leq n-1$,

$$\phi_2(T_{s_i}) = h_{s_i} - \Delta^{-2}(q - q^{-1}) \sum_{c_1 < c_2} F_{c_1}(w_i) F_{c_2}(w_{i+1}),$$

and its inverse ψ_2 is defined as follows:

for $1 \leq j \leq n$,

$$\psi_2(w_j) = t_j,$$

for $i = 0$ or $2 \leq i \leq n-1$,

$$\psi_2(h_{s_i}) = T_{s_i} + \Delta^{-2}(q - q^{-1}) \sum_{c_1 < c_2} F_{c_1}(t_i) F_{c_2}(t_{i+1}).$$

Proof of Theorem 3.2 Combining Propositions 3.3, 3.6 and 3.9, we can see that ϕ preserves the relations involving the generators t_1, \dots, t_n and $T_{s_0}, T_{s_1}, \dots, T_{s_{n-1}}$ in Definition 2.4, and ψ preserves the relations involving the generators w_1, \dots, w_n and $h_{s_0}, h_{s_1}, \dots, h_{s_{n-1}}$ in Definition 3.1, respectively.

By definition, it is obvious that ϕ preserves the relations involving the generators t_1, \dots, t_n and $T_\rho^{\pm 1}$ in Definition 2.4, and ψ preserves the relations involving the generators w_1, \dots, w_n and $h_\rho^{\pm 1}$ in Definition 3.1, respectively. Finally, it suffices to verify that ϕ preserves the relation (2.28) and ψ preserves the relation (3.4), which follows easily from their definitions. \square

Recall that \widehat{W} is the extended affine Weyl group of type A with generators ρ and s_i ($0 \leq i \leq n-1$). For each $\widehat{w} \in \widehat{W}$, let $\widehat{w} = \rho^k s_{i_1} \cdots s_{i_r}$ be a reduced expression of \widehat{w} . From the relations (3.4)-(3.7), we get that $h_{\widehat{w}} := h_\rho^k h_{s_{i_1}} \cdots h_{s_{i_r}}$ is independent of the choice of the reduced expression of \widehat{w} , that is, it is well-defined.

By Theorem 3.2 and Proposition 2.7, we can easily get the following result.

Proposition 3.10. $\widehat{\mathcal{C}}_{r,n}^{\text{aff}}$ has an R -basis consisting of the following elements:

$$\{t_1^{\alpha_1} \cdots t_n^{\alpha_n} h_{\widehat{w}} \mid 0 \leq \alpha_1, \dots, \alpha_n \leq r-1, \widehat{w} \in \widehat{W}\}. \quad (3.50)$$

Proof. From the relations (3.3) and (3.8)-(3.10), we can see that $\widehat{\mathcal{C}}_{r,n}^{\text{aff}}$ is generated over R by the elements of the form $t_1^{\alpha_1} \cdots t_n^{\alpha_n} h_{\widehat{w}}$ with $0 \leq \alpha_i \leq r-1$ and $\widehat{w} \in \widehat{W}$. Thus, it suffices to prove that these elements are linearly independent over R . By (3.3) and (3.8)-(3.10)

again and the morphism ψ defined in Theorem 3.2, we can get that

$$t_1^{\alpha_1} \cdots t_n^{\alpha_n} h_{\widehat{w}} = t_1^{\alpha_1} \cdots t_n^{\alpha_n} T_{\widehat{w}} + \sum_{\substack{\widehat{y} \prec \widehat{w} \\ 0 \leq \beta_i \leq r-1}} t_1^{\beta_1} \cdots t_n^{\beta_n} T_{\widehat{y}}, \quad (3.51)$$

where \prec is the Bruhat order on \widehat{W} . By Proposition 2.7, the set $\{t_1^{\alpha_1} \cdots t_n^{\alpha_n} T_{\widehat{w}} \mid 0 \leq \alpha_i \leq r-1 \text{ and } \widehat{w} \in \widehat{W}\}$ is linearly independent. By (3.51), we then get the desired result. \square

Let $\widehat{\mathcal{H}}_n^{\text{aff}}$ be the R -subalgebra of $\widehat{\mathcal{C}}_{r,n}^{\text{aff}}$ generated by the elements $h_{s_0}, \dots, h_{s_{n-1}}$, and $h_{\rho}^{\pm 1}$, which is canonically isomorphic to the extended affine Hecke algebra of type A , and we shall not distinguish between them. Thus, from Theorems 2.5 and 3.2 we immediately get the following result.

Corollary 3.11. *The extended affine Hecke algebra $\widehat{\mathcal{H}}_n^{\text{aff}}$ of type A is a subalgebra of the affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}$.*

Acknowledgements. The author was partially supported by the National Natural Science Foundation of China (No. 11601273).

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